путем использования телескопических штанг переменной длины с пневматическими приводами. Установлены условия повышения точности станка-робота при обработке осевым инструментом. Они включают соответствие главных осей матрицы жесткости манипулятора и направлению движения осевого инструмента. Определен вид и характер ошибок обработки, и их взаимосвязь с параметрами жесткости станка и манипулятора.

Научная новизна. Впервые разработана концепция высокоточной обработки на станках-роботах, которые имеют низкую жесткость несущей системы, что заключается в применении манипулятора, который жестко закреплен на обрабатываемом объекте и связан с исполнительным органом станка. Впервые установлено необходимое соотношение компонент матриц жесткости станка и манипулятора, которые обеспечивают возможность высокоточной обработки на мобильных станках-роботах. Получили дальнейшее развитие методы расчета погрешностей обработки деталей осевым инструментом на станках с параллельными кинематическими структурами.

Практическая значимость. Результаты исследований являются основой разработки высокоэффективных станков-роботов для обработки опасных объектов в полевых условиях. На основе полученных результатов рабочее пространство мобильного станка-робота увеличивается в 3–5 раз, а точность обработки в полевых условиях повышается до 8–9 квалитетов.

Ключевые слова: *мобильный станок-робот, схемы, модели, жесткость, осевой инструмент, точность, погрешности*

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VARIATION METHOD BASED ON THE INTERPOLATION FOR NAVIER-STOKES SOLUTIONS FOR TRANSIENT UNCOMPRESSIBLE FLOW

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ВАРІАЦІЙНИЙ МЕТОД НА ОСНОВІ ІНТЕРПОЛЯЦІЇ ДЛЯ РІШЕННЯ РІВНЯНЬ НАВ'Є-СТОКСА ДЛЯ НЕСТАЦІОНАРНОЇ НЕСТИСЛИВОЇ ТЕЧІЇ

Purpose. Many studies have been devoted to using variational multiscale (VMS) methods to solve the incompressible flows. The analysis differs when applying the so-called first or second fluctuation operator. On the other hand, VMS methods are used to solve unsteady incompressible flows. Error estimates dependent on the reduced Reynolds number are obtained. On the other hand, the error estimates not dependent on the Reynolds number have already been obtained by using SD and CIP methods. Thus, we desire to obtain the same or similar results by using VMS methods.

Methodology. We propose a fully discrete stabilized method for the unsteady NSEs at high Reynolds number. We use Crank-Nicolson difference in time and use the SV elements in space to preserve the incompressibility. The convective effects are stabilized by adding a new projection-based VMS term. The stability and convergence of the approximation solution are proved. The error estimates hold irrespective of the Reynolds number, and hence also for the incompressible Euler equations, provided the exact solution is smooth.

Findings. We prove the stability and convergence of the approximation solution. The error estimates hold irrespective of the Reynolds number, and hence also for the incompressible Euler equations, provided the exact solution is smooth. This method has good stability. It preserves the incompressibility and it has error estimates not dependent on the viscosity.

Originality. In this paper, we propose a new fully discrete VMS method using SV elements for the unsteady Navier-Stokes at high Reynolds number. Incompressibility is preserved by using Scott-Vogelius elements and convective effects are stabilized by adding a new projection-based variational multiscale (VMS) term.

Practical value. Numerical experiments demonstrate that our method is very effective for incompressible flows at high Reynolds number. They also confirm that our method preserves the incompressibility strongly.

Keywords: *Unsteady Navier-Stokes equations at high Reynolds number, Scott-Vogelius elements, incompressibility, convective effects, Crank-Nicolson difference, varitational multiscale method*

Introduction. The stable and accurate mixed finite element methods for the Navier-Stokes equations (NSEs) at high Reynolds number may suffer from violating LBB (Ladyzhenskaya-Babuška-Brezzi or inf-sup) stability condition and oscillating approximate solutions. The streamline diffusion (SD) methods have been a popular choice to solve these two issues in the past two decades, due to their good stability and high accuracy. They were first proposed by Brooks et al. [1]. Johnson analyzed these methods and extended them to time-dependent problems using time-space elements [2]. He also proposed the SD method on unsteady NSEs based on a stream function-vorticity formulation with divergence-free discrete velocities [3]. Hansbo et al. [4] developed a velocity-pressure SD method using time-space elements for unsteady NSE. More related work on SD methods can be found in [5].

However, the SD methods have some undesirable features such as follows: they introduce additional nonphysical coupling terms between velocity and pressure [6]; they produce inaccuracy of numerical solutions near the boundary; they require calculation of the second derivative when using high order elements and the computational complexity is very big when applied to unsteady Navier-Stokes equations [7].

Recently, alternative stabilized methods have been developed to solve the convection-dominated problems: the variational multiscale methods (VMS), the orthogonal subscales methods, the continuous interior penalty (CIP) methods and the local projection stabilize (LPS) methods [8]. These stabilized methods do not only have good accuracy, but also avoid the undesirable features of SD methods. Both orthogonal subscales methods and LPS methods can be viewed as the special cases of VMS methods, which not only can stabilize the convection term, but also can be used as the LBB stabilization technology for equal-order interpolation [9].

The basic idea of the VMS methods is to define the large scales by projections into appropriate function spaces. Many studies have been devoted to using VMS methods to solve the incompressible flows. The analysis differs when applying the so-called first or second fluctuation operator. On the one hand, VMS methods are used to solve unsteady incompressible flows. Error estimates dependent on the reduced Reynolds number are obtained. On the other hand, the error estimates which are not dependent on the Reynolds number have already been obtained by using SD and CIP methods. Thus, we desire to obtain the same or similar results by using VMS methods. This motivates our work in this paper [10].

Moreover, in the referred works above on the incompressible flows, only the work in enforces the incompressibility strongly (pointwise) by using Scott-Vegelius (SV) elements. The problems that can arise from a poor enforcement of incompressibility are well known even for the simplest problems such as steady NSEs. The SV elements enforce the incompressibility strongly (pointwise), which makes them popular. Moreover, under the restriction that the mesh be a barycenter refinement of a simplex mesh and the degree k of approximating polynomial for velocity is not less than the dimension of the domain space d, i.e. $k \geq d$, the SV elements have recently been shown LBB stable and admit optimal approximation properties. This is also true for the case $d = 3$ and $k = 2$ with a Poweel-Sabin tetrahedral mesh. E. Burman et al. use the LPS and CIP methods with SV elements to solve the steady Oseen problems. Error estimates which are not dependent on the Reynolds number are obtained.

In this paper, we propose a fully discrete stabilized method for the unsteady NSEs at high Reynolds number. We use the Crank-Nicolson difference in time and use the SV elements in space to preserve the incompressibility. The convective effects are stabilized by adding a new projection-based VMS term. The stability and convergence of the approximation solution are proved. The error estimates hold irrespective of the Reynolds number, and hence also for the incompressible Euler equations, provided the exact solution is smooth. Our numerical experiments demonstrate and confirm our theoretical analysis.

We have to stress that our method is different from the VMS methods from these points. The stabilized term in our method applies the fluctuation operator first then applying the gradient operator. The piecewise gradient operator is adopted to allow the projection space to be discontinuous. The analyses of error estimates on our method and other VMS methods are quite different. We obtained an error estimates which hold irrespective of the Reynolds number. This result is almost comparable with the SD and CIP methods. On the other hand, the error estimates are only dependent on a reduced Reynolds number. Moreover, we propose to use SV elements to preserve the incompressibility of the original problem. Also, the approximation of the initial data is taken care of to preserve the incompressibility. In addition, our method is a fully discrete Crank-Nicolson scheme. It is a second order scheme.

An outline of the paper is as follows. In section 2, we introduce necessary notations which will be used in other sections. In section 3, we propose and analyze the stability of our new method. In section 4 we give error estimates for our scheme. In section 5, we give some numerical experiments. In section 6, we conclude the whole paper.

Throughout this paper, we use C to denote a positive constant independent of Δt, h and υ, not necessarily the same at each occurrence.

Basic notations. Let $\Omega \in \mathbb{R}^d$ ($d = 2$ or 3) be a bounded domain with polygonal or polyhedral boundary $\Gamma = \partial\Omega$. We use $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ to denote the m order of Sobolev spaces on Ω, and use $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$ to denote the norm and semi-norm on these spaces. When $p = 2$, we set $H^m(\Omega) = W^{m,p}(\Omega)$, $H_0^m(\Omega) = W_0^{m,p}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,p}$, $|\cdot|_m = |\cdot|_{m,p}$, and the inner product of $H^m(\Omega)$ denoted by $(\cdot,\cdot)_m$, we also let $(\cdot,\cdot) = (\cdot,\cdot)_0$. Let $L_0^2(\Omega) = \left\{ \upsilon \in L^2(\Omega) : \int_\Omega \upsilon \, dx = 0 \right\}$.

Let X denote a Banach space, the mapping $\varphi(x,t):[0,T]\to X$, and we define

$$\|\varphi\|_{L^2(0,T;X)}:=\left(\int_0^T\|\varphi\|_X^2(t)\,dt\right)^{1/2};$$

$$\|\varphi\|_{L^\infty(0,T;X)}:=\sup_{0\le t\le T}\|\varphi\|_X(t).$$

Vector analogues of the Sobolev spaces along with vector-valued functions are denoted by upper and lower case bold face font, respectively, e.g., $\boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{u}$.

Lemma 2.1. Let $v\in H_0^1(\Omega)$, then the Poincaré's inequality holds

$$\|v\|_0\le C_P|v|_1.$$

Let $I=[0,T]$, where T is a positive constant . We consider the Navier-Stokes equations

$$\begin{cases} u_t+u\cdot\nabla u-\upsilon\Delta u+\nabla p=f \text{ in } \Omega\times I\\ \nabla\cdot u=0 \quad\text{in}\quad \Omega\times I\\ u=0 \quad\text{on}\quad \Gamma\times I\\ u=u_0(x)\quad\text{in}\quad\Omega\quad\text{for}\quad t=0\end{cases},\tag{1}$$

where $u=u(x,t)\in\mathbb{R}^d$ denotes the velocity, $p=p(x)\in\mathbb{R}$ denotes the pressure and $f=f(x)\in\mathbb{R}^d$ denotes the body forces, $\upsilon=\mathrm{Re}^{-1}$ denotes the viscosity coefficient. In this paper we consider the situation that Re is a high Reynolds number, i.e. $\upsilon\le Ch$.

Let $V=H_0^1(\Omega)$ and $Q=L_0^2(\Omega)$. A weak formulation of problem (1) is.

We find $(u,p)\in V\times Q$, for all $(v,q)\in V\times Q$ so that

$$(u_t,v)+B(u,p;v,q)+b(u;u,v)=(f,v),$$

where

$$B(u,p;v,q)=\upsilon(\nabla u,\nabla v)-(\nabla\cdot u,q);$$

$$b(w;u,v)=\frac{1}{2}(w\cdot\nabla u,v)-\frac{1}{2}(w\cdot\nabla v,u).$$

Let $\Delta t=T/N$ be the time step, N is a positive integer, $t_n=n\Delta t$, $t_{n+1/2}=(n+1/2)\Delta t$. We use the following

$$f^n:=f(x,t_n),\quad u^n:=u(x,t_n),\quad p^n:=p(x,t_n).$$

We also define

$$u^{n+1/2}=\frac{u^{n+1}+u^n}{2},\quad p^{n+1/2}=\frac{p^{n+1}+p^n}{2}.$$

Then from (1) we have

$$\begin{aligned}&B\left(u^{n+1/2},\frac{p^{n+1}+p^n}{2};v,q\right)+\left(\frac{u^{n+1}-u^n}{\Delta t},v\right)+\\&+b\left(u^{n+1/2};u^{n+1/2},v\right)=\left(\frac{u^{n+1}-u^n}{\Delta t}-u_t\left(x,t_{n+1/2}\right),v\right)+\\&+B\left(u^{n+1/2}-u\left(x,t_{n+1/2}\right),p^{n+1/2}-p\left(x,t_{n+1/2}\right);v,q\right)+\\&+\left(b\left(u^{n+1/2};u^{n+1/2},v\right)-b\left(u\left(x,t_{n+1/2}\right);u\left(x,t_{n+1/2}\right),v\right)\right)+\\&+\left(f\left(x,t_{n+1/2}\right),v\right):=R^{n+1}(v)+\left(f\left(x,t_{n+1/2}\right),v\right).\end{aligned}\tag{2}$$

The following lemma is trivial.

Lemma 2.2. Suppose $w_{tt}, w_{ttt}\in L^2(0,T;L^2(\Omega))$, we have the estimates

$$\left\|w_t\left(x,t_{n+1/2}\right)-\frac{w^{n+1}-w^n}{\Delta t}\right\|_0^2\le C\Delta t^3\|w_{ttt}\|^2_{L^2\left(t_n,t_{n+1};L^2(\Omega)\right)};$$

$$\left\|w^{n+1/2}-w\left(x,t_{n+1/2}\right)\right\|_0^2\le C\Delta t^3\|w_{tt}\|^2_{L^2\left(t_n,t_{n+1};L^2(\Omega)\right)}.$$

Using Lemma 2.1 we can give an estimate for $R^{n+1}(v)$.

By corollary 3, we assume that u is satisfying

$$u_{ttt}\in L^2(0,T;L^2(\Omega));\quad u_{tt}\in L^2(0,T;H^2(\Omega));$$

$$p_{tt}\in L^2(0,T;H^1(\Omega));\quad u\in L^\infty(0,T;W^{1,\infty}(\Omega)),$$

then we have the estimates for $R^{n+1}(v)$

$$\left|R^{n+1}(v)\right|\le C\Delta t^{3/2}\|v\|_0\left(1+\left\|u^{n+1/2}\right\|_{1,\infty}\right)\times$$

$$\times\left(\|u_{ttt}\|_{L^2\left(t_n,t_{n+1};L^2(\Omega)\right)}+\|u_{tt}\|_{L^2\left(t_n,t_{n+1};H^2(\Omega)\right)}+\|p_{tt}\|_{L^2\left(t_n,t_{n+1};H^1(\Omega)\right)}\right).$$

Let $\mathcal{J}=\{K\}$ be a barycenter refine mesh of a quasi-uniform triangular or tetrahedral mesh. For all $K\in\mathcal{J}_h$, let h_K be the diameter of K and $h=\max_{K\in\mathcal{J}_h}h_K$. Let $P_{k+1}(\mathcal{J}_h)$ represent the $k+1$-order continuous piecewise polynomial on decomposition $\mathcal{J}_h$, and $P_k^{dc}(\mathcal{J}_h)$ represent the k-order discontinuous piecewise polynomial on decomposition $\mathcal{J}_h$, where $k\ge 0$ is an integer; let $P_k(K)$, $k\ge 0$ be the k-order polynomial on K; let $P_1^{nc}(\mathcal{J}_h)$ represent the nonconforming linear element on decomposition $\mathcal{J}_h$.

We define the r-order SV elements ($r\ge 2$) $V_h\times Q_h\subset V\times Q$ as

$$V_h=H_0^1(\Omega)\cap P_r(\mathcal{J}_h);\quad Q_h=L_0^2(\Omega)\cap P_{r-1}^{dc}(\mathcal{J}_h).$$

Notice when $r=2$, $d=3$, the mesh $\mathcal{J}_h$ is constructed by a Powell-Sabin tetrahedralizeation . We also introduce the divergence-free space of V_h

$$W_h=\{w_h\in V_h:(\nabla\cdot w_h,q_h)=0,\quad\forall q_h\in Q_h\}.\ (3)$$

Clearly, since we are using the SV elements, we can choose $q_h=\nabla\cdot w_h$, then the definition of W_h is equivalent to

$$W_h=\{w_h\in V_h:\nabla\cdot w_h|_K=0,\quad\forall K\in\mathcal{J}_h\}.$$

Then we have the following conclusions.

Lemma 2.4. For every $(u,p)\in\left(H_0^1(\Omega)\cap H^{r+1}(\Omega)\right)\times\left(L_0^2(\Omega)\cap H^r(\Omega)\right)$, there exist two interplants $I_h:H_0^1(\Omega)\to V_h$ and $J_h:L_0^2(\Omega)\to Q_h$ so that

$$\|u-I_hu\|_0+h|u-I_hu|_1\le Ch^{r+1}\|u\|_{r+1},$$

when $u\in W^{1,\infty}(\Omega)$,

$$\|u+I_hu\|_{0,\infty}+h|u-I_hu|_{1,\infty}\le Ch\|u\|_{1,\infty},$$

when $\nabla\cdot u=0$ and $u\in H_0^1(\Omega)\cap H^{r+1}(\Omega)$

$$I_h u \in W_h;$$
$$\|p - J_h p\|_0 + h|p - J_h p|_{1,h} \le Ch^r \|p\|_r;$$
$$(p - J_h p, \nabla \cdot \upsilon_h) = 0 \forall \upsilon_h \in V_h.$$

Where $\left|p - J_h p\right|_{1,h}^2 = \sum_{K \in \mathcal{J}_h} \left|p - J_h p\right|_{1,K}^2$.

We recall the following inverse estimate.

Lemma 2.5. For all $\upsilon_h \in V_h$ and $K \in \mathcal{J}_h$, $0 < h \le 1$, $0 \le m \le l$, $1 \le p$, $q \le \infty$, there exists a positive constant C_{inv} independent of h, K, p and q so that

$$\left|\upsilon_h\right|_{l,p,K} \le C_{\text{inv}} h_K^{m-l+d\left(\frac{1}{p}-\frac{1}{q}\right)} \left|\upsilon_h\right|_{m,q,K}.$$

A new projection-based VMS method. ***Fully discrete scheme.*** Let L_H be a l-order ($l \le r$) finite element defined on a mesh $\mathcal{J}_H$ ($\mathcal{J}_H = \mathcal{J}_h$ is acceptable and L_H can be discontinuous) and $L_H \ne V_H$, let P_H be a well-defined mapping from $L^2(\Omega)$ to $L_H\, L_H$ satisfying

$$\forall u \in H^{k+1}(\Omega), \quad 0 \le k \le 1$$
$$|(id - P_H)u|1 \le CH^k \|u\|_{k+1}, \tag{4}$$

where $id : L^2(\Omega) \to L^2(\Omega)$ is an identity. Let ∇_H be the piecewise gradient operator defined on the mesh $\mathcal{J}_H$. We consider a projection-based VMS method for (1). We use SV elements FEM in space and Crank-Nicolson difference in time. The discretization reads.

We find $\left(u_h^{n+1}, p_h^{n+1}\right) \in V_h \times Q_h$, for all $(\upsilon_h, q_h) \in V_h \times Q_h$ so that

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, \upsilon_h\right) + b\left(u_h^{n+1/2}; u_h^{n+1/2}, \upsilon_h\right) +$$
$$+ B\left(u_h^{n+1/2}, p_h^{n+1/2}; \upsilon_h, q_h\right) + S_h\left(u_h^{n+1/2}, \upsilon_h\right) = \tag{5}$$
$$= \left(f\left(x, t_{n+1/2}\right), \upsilon_h\right),$$

where $n = 0, 1, 2, \ldots, N-1$ and

$$S_h(u, \upsilon) = \alpha(\nabla_H(id - P_H)u, \nabla_H(id - P_H)\upsilon). \tag{6}$$

To preserve the incompressible property, the initial level u_h^0 is approximated by finding $\left(u_h^0, \lambda_h^0\right) \in V_h \times Q_h$ so that

$$\left(u_h^0, \upsilon_h\right) + B\left(u_h^0, \lambda_h^0; \upsilon_h, q_h\right) =$$
$$= \left(u_0, \upsilon_h\right) + B\left(u_0, \lambda_0; \upsilon_h, q_h\right), \quad \forall \left(\upsilon_h, q_h\right) \in V_h \times Q_h,$$

where $\lambda_0 = 0$. From (6, 4) and Lemma 2.4 it is easy to see that

$$\left\|u_h^0\right\|_0^2 + \upsilon \left\|\nabla u_h^0\right\|_0^2 \le \left\|u_0\right\|_0^2 + \upsilon \left\|\nabla u_0\right\|_0^2; \tag{7}$$

$$\left\|u_h^0 - u_0\right\|_0^2 + \upsilon \left\|\nabla\left(u_h^0 - u_0\right)\right\|_0^2 \le C\left(h^{2r+2} + \upsilon h^{2r}\right)\left\|u_0\right\|_{r+1}^2. \tag{8}$$

To derive the error estimates, we consider the following equivalent problem: find $u_h^n \in W_h$ so that for all $\upsilon_h \in W_h$, there holds

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, \upsilon_h\right) + b\left(u_h^{n+1/2}; u_h^{n+1/2}, \upsilon_h\right) + \upsilon\left(\nabla u_h^{n+1/2}, \nabla \upsilon_h\right) +$$
$$+ S_h\left(u_h^{n+1/2}, \upsilon_h\right) = \left(f\left(x, t_{n+1/2}\right), \upsilon_h\right). \tag{9}$$

Remark. The term $S_h(u, \upsilon)$ in our method is applying the fluctuation operator first then applying the gradient operator, which is different from the VMS methods. Moreover, the analyses of the error estimates in our method and papers are quite different. Thus, we obtain error estimates which are not dependent on the Reynolds number.

Stability. In this sub-section, we prove the stability of the velocities as follows.

Theorem 3.1. Let $\left(u_h^{n+1}, p_h^{n+1}\right) \in V_h \times Q_h$ denote the solution of (5), then for any arbitrary positive constant C_0, when $\Delta t < C_0$, we have the following results

$$\max_{0 \le i \le N-1} \left\|u_h^{i+1}\right\|_0^2 + 2\Delta t \sum_{i=0}^{N-1} \left(\upsilon \left\|\nabla u_h^{i+1/2}\right\|_0^2 + S_h\left(u_h^{i+1/2}, u_h^{i+1/2}\right)\right) \le$$
$$\le \exp\left(\frac{T}{C_0 - \Delta t}\right) \times \tag{10}$$
$$\times \left[C_0 \Delta t \sum_{i=0}^{N-1} \left\|f\left(x, t_{i+1/2}\right)\right\|_0^2 + \left\|u^0\right\|_0^2 + \upsilon \left\|\nabla u_0\right\|_0^2\right].$$

Proof. Testing (5) with $\left(\upsilon_h, q_h\right) = \left(u_h^{n+1/2}, p_h^{n+1/2}\right)$ gives

$$\frac{1}{2\Delta t}\left(\left\|u_h^{n+1}\right\|_0^2 - \left\|u_h^n\right\|_0^2\right) + \upsilon \left\|\nabla u_h^{n+1/2}\right\|_0^2 + S_h\left(u_h^{n+1/2}, u_h^{n+1/2}\right) =$$
$$= \left(f\left(x, t_{n+1/2}\right), u_h^{n+1/2}\right) \le \frac{C_0}{2}\left\|f\left(x, t_{n+1/2}\right)\right\|_0^2 + \frac{1}{2C_0}\left\|u_h^{n+1/2}\right\|_0^2 \le$$
$$\le \frac{C_0}{2}\left\|f\left(x, t_{n+1/2}\right)\right\|_0^2 + \frac{1}{4C_0}\left(\left\|u_h^{n+1}\right\|_0^2 + \left\|u_h^n\right\|_0^2\right).$$

This becomes

$$\left\|u_h^{n+1}\right\|_0^2 - \left\|u_h^n\right\|_0^2 + 2\Delta t \upsilon \left\|\nabla \frac{u_h^{n+1} + u_h^n}{2}\right\|_0^2 + 2\Delta t S_h\left(u_h^{n+1/2}, u_h^{n+1/2}\right) \le$$
$$\le C_0 \Delta t \left\|f\left(x, t_{n+1/2}\right)\right\|_0^2 + \frac{\Delta t}{2C_0}\left(\left\|u_h^{n+1}\right\|_0^2 + \left\|u_h^n\right\|_0^2\right).$$

Adding the last inequation from 0 to and using (7) we get

$$\left\|u_h^{n+1}\right\|_0^2 + 2\Delta t \sum_{i=0}^{n} \left(\upsilon \left\|\nabla u_h^{i+1/2}\right\|_0^2 + S_h\left(u_h^{n+1/2}, u_h^{n+1/2}\right)\right) \le$$
$$\le C_0 \Delta t \sum_{i=0}^{n} \left\|f\left(x, t_{i+1/2}\right)\right\|_0^2 + \frac{\Delta t}{2C_0}\left\|u_h^{n+1}\right\|_0^2 +$$
$$+ \frac{\Delta t}{C_0} \sum_{i=1}^{n} \left\|u_h^i\right\|_0^2 + \frac{\Delta t}{2C_0}\left\|u_h^0\right\|_0^2 + \left\|u_h^0\right\|_0^2 \le$$
$$\le C_0 \Delta t \sum_{i=0}^{n} \left\|f\left(x, t_{i+1/2}\right)\right\|_0^2 + \sum_{i=0}^{n+1} \left\|u_h^i\right\|_0^2 + \left\|u^0\right\|_0^2 + \upsilon \left\|\nabla u_0\right\|_0^2,$$

when $\frac{\Delta t}{C_0} < 1$, by discrete Gronwall's inequality we have

$$\left\|u_h^{n+1}\right\|_0^2+2\Delta t\sum_{i=0}^{n}\left(\upsilon\left\|\nabla u_h^{i+1/2}\right\|_0^2+S_h\left(u_h^{n+1/2},u_h^{n+1/2}\right)\right)\le$$
$$\le\exp\left(\Delta t\sum_{i=0}^{n+1}\frac{C_0^{-1}}{1-\Delta tC_0^{-1}}\right)\times$$
$$\times\left[C_0\Delta t\sum_{i=0}^{n}\left\|f\left(x,t_{i+1/2}\right)\right\|_0^2+\left\|u^0\right\|_0^2+\upsilon\left\|\nabla u_0\right\|_0^2\right]\le$$
$$\le\exp\left(\frac{T}{C_0-\Delta t}\right)\cdot\left[C_0\Delta t\sum_{i=0}^{n}\left\|f\left(x,t_{i+\frac{1}{2}}\right)\right\|_0^2+\left\|u^0\right\|_0^2+\upsilon\left\|\nabla u_0\right\|_0^2\right],$$

this proves (10).

Priori error estimates. In this section, we always assume that the solution (u, p) of equation (1) satisfies the following regular condition, there is a constant C independent of h, υ and Δt so that

$$\max\left\{\begin{matrix}\|u\|_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)},\|u\|_{L^\infty\left(0,T;H^{r+1}(\Omega)\right)}\\ \|u_t\|_{L^2\left(0,T;H^{r+1}(\Omega)\right)}\end{matrix}\right\}+$$
$$+\max\left\{\begin{matrix}\|u_{ttt}\|_{L^2\left(0,T;L^2(\Omega)\right)},\|u_{tt}\|_{L^2\left(0,T;H^2(\Omega)\right)}\\ \|p_{tt}\|_{L^2\left(0,T;H^1(\Omega)\right)},\|p\|_{L^\infty\left(0,T;H^{r+1}(\Omega)\right)}\end{matrix}\right\}\le C.$$

Let $(u^{n+1}, p^{n+1})\in V\times Q$ denote the solution of (1) at time level t_{n+1} and $\left(u_h^{n+1},p_h^{n+1}\right)\in V_h\times Q_h$ denote the solution of (5), and let I_h and J_h be as stated in Lemma 2.4. For simplicity, we use the following definitions

$$\left(e_u^n,e_p^n\right):=\left(u^n-u_h^n,p^n-p_h^n\right); \qquad (11)$$
$$\left(\eta^n,\beta^n\right):=\left(u^n-I_hu^n,p^n-J_hp^n\right); \qquad (12)$$
$$\left(\xi^n,\gamma^n\right):=\left(I_hu^n-u_h^n,J_hp^n-p_h^n\right). \qquad (13)$$

The following conclusion is given to prepare the proof of error estimates.

Lemma 4.1. With the use of the definitions above, there always holds

$$\left\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\right\|_0^2\le\frac{Ch^{2r+2}}{\Delta t}\|u_t\|^2_{L^2\left(t_n,t_{n+1};H^{r+1}(\Omega)\right)}. \qquad (14)$$

Now we can give an error estimate for the velocity.

Theorem 4.2 (Velocity error estimate). Let $(u^{n+1}, p^{n+1})\in V\times Q$ denote the solution of (1) at time level t_{n+1} and $\left(u_h^{n+1},p_h^{n+1}\right)\in V_h\times Q_h$ denote the solution of (5), when $h\le 1$, $\Delta tC_t<1$, there holds

$$\max_{1\le i\le N}\left\|u^i-u_h^i\right\|_0^2+\Delta t\sum_{i=0}^{N-1}\upsilon\left\|\nabla\left(u^{i+1/2}-u_h^{i+1/2}\right)\right\|_0^2+$$
$$+\Delta t\sum_{i=0}^{N-1}S_h\left(u^{i+1/2}-u_h^{i+1/2},u^{i+1/2}-u_h^{i+1/2}\right)\le$$
$$\le C_{\exp}C_{E_1}\left(\upsilon h^{2r}+\alpha h^{2r}+\alpha^{-1}h^{2r+2}+H^{-2}h^{2r+2}+\alpha H^{2l}\right)+$$
$$+C_{\exp}C_{E_2}\Delta t^4,$$

where

$$C_{E_1}=C\left(\|u\|^2_{L^\infty\left(0,T;H^{r+1}(\Omega)\right)}\|u\|^2_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)}+\|u\|^2_{L^\infty\left(0,T;H^{r+1}(\Omega)\right)}\right)+$$
$$+C\|u_t\|^2_{L^2\left(0,T;H^{r+1}(\Omega)\right)};$$
$$C_{E_2}=C\left(\|u_{ttt}\|^2_{L^2\left(0,T;L^2(\Omega)\right)}+\|u_{tt}\|^2_{L^2\left(0,T;H^2(\Omega)\right)}+\|p_{tt}\|^2_{L^2\left(0,T;H^1(\Omega)\right)}\right);$$
$$C_t=C\left(\|u\|^2_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)}+\|u\|_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)}+1\right);$$
$$C_{\exp}=\exp\left(\frac{C_tT}{1-\Delta tC_t}\right).$$

Proof. Let us take $\upsilon_h\in W_h$, then subtracting (9) from (2) gives

$$\left(\frac{e_u^{n+1}-e_u^n}{\Delta t},\upsilon_h\right)+\upsilon\left(\nabla e_u^{n+1/2},\nabla\upsilon_h\right)+S_h\left(e_u^{n+1/2},\upsilon_h\right)+$$
$$+\left(b\left(u^{n+1/2};u^{n+1/2},\upsilon_h\right)-b\left(u_h^{n+1/2};u_h^{n+1/2},\upsilon_h\right)\right)=$$
$$=S_h\left(u^{n+1/2},\upsilon_h\right)+R^{n+1}\left(\upsilon_h\right).$$

Using (11–13) we rewrite the last equality as

$$\begin{aligned}&\left(\frac{\xi^{n+1}-\xi^n}{\Delta t},\upsilon_h\right)+\upsilon\left(\nabla\xi^{n+1/2},\nabla\upsilon_h\right)+S_h\left(\xi^{n+1/2},\upsilon_h\right)=\\&=-\left(\frac{\eta^{n+1}-\eta^n}{\Delta t},\upsilon_h\right)+S_h\left(u^{n+1/2},\upsilon_h\right)+R^{n+1}\left(\upsilon_h\right)-\\&-\upsilon\left(\nabla\eta^{n+1/2},\nabla\upsilon_h\right)-S_h\left(\eta^{n+1/2},\upsilon_h\right)+\\&+\left(b\left(u_h^{n+1/2};u_h^{n+1/2},\upsilon_h\right)-b\left(u^{n+1/2};u^{n+1/2},\upsilon_h\right)\right).\end{aligned} \qquad (15)$$

Testing (15) with $\upsilon_h=\xi^{n+1/2}$ gives

$$\begin{aligned}&\frac{1}{2\Delta t}\left(\left\|\xi^{n+1}\right\|_0^2-\left\|\xi^n\right\|_0^2\right)+\upsilon\left\|\nabla\xi^{n+1/2}\right\|_0^2+S_h\left(\xi^{n+1/2},\xi^{n+1/2}\right)=\\&=-\left(\frac{\eta^{n+1}-\eta^n}{\Delta t},\xi^{n+1/2}\right)-\upsilon\left(\eta^{n+1/2},\xi^{n+1/2}\right)-\\&-S_h\left(\eta^{n+1/2},\xi^{n+1/2}\right)+\left(b\left(u_h^{n+1/2};u_h^{n+1/2},\xi^{n+1/2}\right)-\right.\\&\left.-b\left(u^{n+1/2};u^{n+1/2},\xi^{n+1/2}\right)\right)+S_h\left(u^{n+1/2},\xi^{n+1/2}\right)+\\&+R^{n+1}\left(\xi^{n+1/2}\right)=:\sum_{i=1}^{6}R_i.\end{aligned} \qquad (16)$$

Now we estimate R_i term by term. Using (14) we get

$$\left|R_1\right|\le\left\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\right\|_0\left\|\xi^{n+1/2}\right\|_0\le$$
$$\le Ch^{2r+2}\Delta t^{-1}\|u_t\|^2_{L^2\left(t_n,t_{n+1};H^{r+1}(\Omega)\right)}+\left\|\xi^{n+1/2}\right\|_0^2;$$
$$\left|R_2\right|\le\left|\upsilon\left(\nabla\eta^{n+1/2},\nabla\xi^{n+1/2}\right)\right|\le C\upsilon h^r\left\|u^{n+1/2}\right\|_{r+1}\left\|\nabla\xi^{n+1/2}\right\|_0\le$$
$$\le C\upsilon h^{2r}\left\|u^{n+1/2}\right\|_{r+1}^2+\frac{\upsilon}{2}\left\|\nabla\xi^{n+1/2}\right\|_0^2.$$

By the definition of Sh(·;·), (4) with $k = 0$ and (3), there holds

$$\begin{aligned}|R_3| &\le \left(S_h\left(\eta^{n+1/2},\eta^{n+1/2}\right)S_h\left(\xi^{n+1/2},\xi^{n+1/2}\right)\right)^{1/2} \le \\ &\le 4S_h\left(\eta^{n+1/2};\eta^{n+1/2}\right)+\frac{1}{16}S_h\left(\xi^{n+1/2};\xi^{n+1/2}\right) \le \\ &\le C\alpha h^{2r}\left\|u^{n+1/2}\right\|_{r+1}^2+\frac{1}{16}S_h\left(\xi^{n+1/2};\xi^{n+1/2}\right).\end{aligned}$$

Using triangle inequality, Green's formula, $\nabla\cdot u=\nabla\cdot u_h^{n+1/2}=\nabla\cdot I_h u_h^{n+1/2}=0$ and the fact that $b(w; v, v)$, we estimate R_4 as

$$\begin{aligned}|R_4| &\le \left|\left(\xi^{n+1/2}\cdot\nabla\xi^{n+1/2},\eta^{n+1/2}\right)\right|+\left|\left(\eta^{n+1/2}\cdot\nabla\eta^{n+1/2},\xi^{n+1/2}\right)\right|+ \\ &+\left|\left(u^{n+1/2}\cdot\nabla\eta^{n+1/2},\xi^{n+1/2}\right)\right|+\left|\left(\xi^{n+1/2}\cdot\nabla u^{n+1/2},\xi^{n+1/2}\right)\right|+ \\ &+\left|\left(\eta^{n+1/2}\cdot\nabla u^{n+1/2},\xi^{n+1/2}\right)\right|:=\sum_{i=1}^{5}R_{4,i}.\end{aligned}$$

Using Hölder's inequality, inverse inequality, we obtain

$$\begin{aligned}&\left|R_{4,1}\right| \le \left|\left(\xi^{n+1/2}\cdot\nabla\xi^{n+1/2},\eta^{n+1/2}\right)\right| \le \\ &\le C\left\|\xi^{n+1/2}\right\|_0\left\|\nabla\xi^{n+1/2}\right\|_0\left\|\eta^{n+1/2}\right\|_{0,\infty} \le C\left\|u^{n+1/2}\right\|_{1,\infty}\left\|\xi^{n+1/2}\right\|_0^2; \\ &\left|R_{4,2}\right| \le \left\|\eta^{n+1/2}\right\|_{0,\infty}\left\|\nabla\eta^{n+1/2}\right\|_0\left\|\xi^{n+1/2}\right\|_0 \le \\ &\le Ch^{r+1}\left\|u^{n+1/2}\right\|_{r+1}\left\|u^{n+1/2}\right\|_{1,\infty}\left\|\xi^{n+1/2}\right\|_0 \le \\ &\le Ch^{2r+2}\left\|u^{n+1/2}\right\|_{r+1}^2\left\|u^{n+1/2}\right\|_{1,\infty}^2+\left\|\xi^{n+1/2}\right\|_0^2.\end{aligned}$$

Using triangle inequality, inverse inequality and L^2-stability of P_H, we have

$$\begin{aligned}\left|R_{4,3}\right| &= \left|\left(u^{n+1/2}\cdot\nabla\xi^{n+1/2},\eta^{n+1/2}\right)\right| \le \\ &\le \left|\left(u^{n+1/2}\cdot\nabla_H P_H\xi^{n+1/2},\eta^{n+1/2}\right)\right|+ \\ &+\left|\left(u^{n+1/2}\cdot\nabla_H\left(id-P_H\right)\xi^{n+1/2},\eta^{n+1/2}\right)\right| \le \\ &\le \left\|u^{n+1/2}\right\|_{0,\infty}\left\|\nabla_H P_H\xi^{n+1/2}\right\|_0\left\|\eta^{n+1/2}\right\|_0+ \\ &+\left\|u^{n+1/2}\right\|_{0,\infty}\left\|\nabla_H\left(id-P_H\right)\xi^{n+1/2}\right\|_0\left\|\eta^{n+1/2}\right\|_0 \le \\ &\le Ch^{r+1}H^{-1}\left\|u^{n+1/2}\right\|_{0,\infty}\left\|u^{n+1/2}\right\|_{r+1}\left\|\xi^{n+1/2}\right\|_0+ \\ &+Ch^{r+1}\left\|u^{n+1/2}\right\|_{0,\infty}\left\|u^{n+1/2}\right\|_{r+1}\left\|\nabla_H\left(id-P_H\right)\xi^{n+1/2}\right\|_0 \le \\ &\le Ch^{2r}\left(H^{-2}h^2+\alpha^{-1}h^2\right)\left\|u^{n+1/2}\right\|_{r+1}^2\left\|u^{n+1/2}\right\|_{1,\infty}^2+ \\ &+\left\|\xi^{n+1/2}\right\|_0^2+\frac{1}{16}S_h\left(\xi^{n+1/2};\xi^{n+1/2}\right).\end{aligned}$$

Using Hölder's inequality, inverse inequality, one gets

$$\begin{aligned}\left|R_{4,4}\right|+\left|R_{4,5}\right| &\le \left\|\nabla u^{n+1/2}\right\|_{0,\infty}\left\|\xi^{n+1/2}\right\|_0^2+ \\ &+\left\|\eta^{n+1/2}\right\|_0\left\|\nabla u^{n+1/2}\right\|_{0,\infty}\left\|\xi^{n+1/2}\right\|_0 \le \\ &\le Ch^{2r+2}\left\|u^{n+1/2}\right\|_{r+1}^2\left\|u^{n+1/2}\right\|_{1,\infty}^2+\left(\left\|\nabla u^{n+1/2}\right\|_{0,\infty}+1\right)\left\|\xi^{n+1/2}\right\|_0^2; \\ \left|R_5\right| &\le 4S_h\left(u^{n+1/2},u^{n+1/2}\right)+\frac{1}{16}S_h\left(\xi^{n+1/2},\xi^{n+1/2}\right) \le \\ &\le C\alpha H^{2l}\left\|u^{n+1/2}\right\|_{l+1}^2+\frac{1}{16}S_h\left(\xi^{n+1/2},\xi^{n+1/2}\right).\end{aligned}$$

By Corollary 3 we have

$$\begin{aligned}\left|R_6\right| &\le \left(1+\left\|u^{n+1/2}\right\|_{1,\infty}^2\right)\left\|\xi^{n+1/2}\right\|_0^2+ \\ &+C\Delta t^3\left(\left\|u_{ttt}\right\|_{L^2\left(t_n,t_{n+1};L^2(\Omega)\right)}^2+\left\|u_{tt}\right\|_{L^2\left(t_n,t_{n+1};H^2(\Omega)\right)}^2+\right. \\ &\left.+\left\|p_{tt}\right\|_{L^2\left(t_n,t_{n+1};H^1(\Omega)\right)}^2\right).\end{aligned} \tag{17}$$

From (16) to (17) we get

$$\begin{aligned}&\frac{1}{2\Delta t}\left(\left\|\xi^{n+1}\right\|_0^2-\left\|\xi^n\right\|_0^2\right)+\frac{\upsilon}{2}\left\|\nabla\xi^{n+1}\right\|_0^2+\frac{1}{2}S_h\left(\xi_h^{n+1},\xi_h^{n+1}\right) \le \\ &\le C\Delta t^3\times\left(\left\|u_{ttt}\right\|_{L^2\left(t_n,t_{n+1};L^2(\Omega)\right)}^2+\right. \\ &\left.+\left\|u_{tt}\right\|_{L^2\left(t_n,t_{n+1};H^2(\Omega)\right)}^2+\left\|p_{tt}\right\|_{L^2\left(t_n,t_{n+1};H^1(\Omega)\right)}^2\right)+ \\ &+Ch^{2r+2}\Delta t^{-1}\left\|u_t\right\|_{L^2\left(t_n,t_{n+1};H^{r+1}(\Omega)\right)}^2+ \\ &+C\left(\left\|u^{n+1/2}\right\|_{1,\infty}^2+\left\|u^{n+1/2}\right\|_{1,\infty}+1\right)\left(\left\|\xi^{n+1}\right\|_0^2+\left\|\xi^n\right\|_0^2\right)+ \\ &+C\left(\upsilon h^{2r}+\alpha h^{2r}+\alpha^{-1}h^{2r+2}+H^{-2}h^{2r+2}+\alpha H^{2l}\right)\times \\ &\times\left(\left\|u^{n+1/2}\right\|_{r+1}^2\left\|u^{n+1/2}\right\|_{1,\infty}^2+\left\|u^{n+1/2}\right\|_{r+1}^2\right).\end{aligned} \tag{18}$$

We add (18) form 0 to n to get

$$\begin{aligned}&\sum_{i=0}^{n}\Delta t\left(\upsilon\left\|\nabla\xi^{i+1}\right\|_0^2+S_h\left(\xi^{i+1},\gamma^{i+1};\xi^{i+1},\gamma^{i+1}\right)\right)+\left\|\xi^{n+1}\right\|_0^2 \le \\ &\le C_{E_1}\left(\upsilon h^{2r}+\alpha h^{2r}+\alpha^{-1}h^{2r+2}+H^{-2}h^{2r+2}+\alpha H^{2l}\right)+ \\ &+C_{E_2}\Delta t^4+C_t\sum_{i=0}^{n}\Delta t\left\|\xi^{i+1}\right\|_0^2+\left\|\xi^0\right\|_0^2,\end{aligned}$$

where

$$\begin{aligned}C_{E_1} &= C\left(\left\|u\right\|_{L^\infty\left(0,T;H^{r+1}(\Omega)\right)}^2\left\|u\right\|_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)}^2+\left\|u\right\|_{L^\infty\left(0,T;H^{r+1}(\Omega)\right)}^2\right)+ \\ &+C\left\|u_t\right\|_{L^2\left(0,T;H^{r+1}(\Omega)\right)}^2; \\ C_{E_2} &= C\left(\left\|u_{ttt}\right\|_{L^2\left(0,T;L^2(\Omega)\right)}^2+\left\|u_{tt}\right\|_{L^2\left(0,T;H^2(\Omega)\right)}^2+\left\|p_{tt}\right\|_{L^2\left(0,T;H^1(\Omega)\right)}^2\right); \\ C_{E_2} &= C\left(\left\|u_{ttt}\right\|_{L^2\left(0,T;L^2(\Omega)\right)}^2+\left\|u_{tt}\right\|_{L^2\left(0,T;H^2(\Omega)\right)}^2+\left\|p_{tt}\right\|_{L^2\left(0,T;H^1(\Omega)\right)}^2\right).\end{aligned}$$

By discrete Gronwall's inequality and (8), when $\Delta tC_t < 1$, we have

$$\left\|\xi^{n+1}\right\|_0^2+\sum_{i=0}^{n}\Delta t\left(\upsilon\left\|\nabla\xi^{i+1}\right\|_0^2+S_h\left(\xi^{i+1},\gamma^{i+1};\xi^{i+1},\gamma^{i+1}\right)\right)\le$$
$$\le C_{\exp}C_{E_1}\left(\upsilon h^{2r}+\alpha h^{2r}+\alpha^{-1}h^{2r+2}+H^{-2}h^{2r+2}+\alpha H^{2l}\right)+$$
$$+C_{\exp}C_{E_2}\Delta t^4,$$

where $C_{\exp}=\exp\left(\dfrac{C_tT}{1-\Delta tC_t}\right)$.

By triangle inequality we get our final conclusion.

Theorem 4.3 (Pressure error estimate). Under the condition of Theorem 4.2, we have

$$\Delta t\sum_{i=0}^{N-1}\left\|p^{i+1/2}-p_h^{i+1/2}\right\|_0^2\le\Delta t\sum_{i=0}^{N-1}\left(\left\|\frac{e_u^{i+1}-e_u^i}{\Delta t}\right\|_0^2+\upsilon\left\|\nabla e_u^{i+1/2}\right\|_0^2\right)+$$
$$+C\Delta t\sum_{i=0}^{N-1}\left(\left(\max\left(\left\|u^{i+1/2}\right\|_{0,\infty},\left\|u_h^{i+1/2}\right\|_{0,\infty}^2\right)\left\|e_u^{i+1/2}\right\|_0^2\right)+\right.$$
$$\left.+\alpha\left\|\nabla_H\left(id-P_H\right)e_u^{i+1/2}\right\|_0^2+\alpha\left\|\nabla_H\left(id-P_H\right)u^{i+1/2}\right\|_0^2\right)+$$
$$+\Delta t\sum_{i=0}^{N-1}\left\|p^{i+1/2}-J_hp^{i+1/2}\right\|_0^2.$$

Proof. Subtracting (5) form (2) gives

$$\left(\frac{e_u^{n+1}-e_u^n}{\Delta t},\upsilon_h\right)+B\left(e_u^{n+1/2},e_p^{n+1/2};\upsilon_h,q_h\right)+$$
$$+\left(b\left(u^{n+1/2};u^{n+1/2},\upsilon_h\right)-b\left(u_h^{n+1/2};u_h^{n+1/2},\upsilon_h\right)\right)+\qquad(19)$$
$$+S_h\left(e_u^{n+1/2},\upsilon_h\right)=S_h\left(u^{n+1/2},\upsilon_h\right)+R^{n+1}\left(\upsilon_h\right).$$

Test (19) with $q_h=0$, use $\nabla\cdot u=\nabla\cdot u_h^{n+1/2}=0$ and rearrange to get

$$-\left(\nabla\cdot\upsilon_h,\gamma^{n+1/2}\right)=-\left(\frac{e_u^{n+1}-e_u^n}{\Delta t},\upsilon_h\right)-\upsilon\left(\nabla e_u^{n+1/2},\nabla\upsilon_h\right)+$$
$$+\left(\nabla\cdot\upsilon_h,\beta^{n+1/2}\right)+\left(b\left(u_h^{n+1/2};u_h^{n+1/2},\upsilon_h\right)-b\left(u^{n+1/2};u^{n+1/2},\upsilon_h\right)\right)-$$
$$-S_h\left(e_u^{n+1/2},\upsilon_h\right)+S_h\left(u^{n+1/2},\upsilon_h\right)+R^{n+1}\left(\upsilon_h\right)=:\sum_{i=1}^{7}R_i.$$

Estimate R_i term by term. Using Poincare's inequality and $(\nabla\cdot\upsilon_h,\beta^{n+1/2})$, we get

$$\left|R_1+R_2+R_3\right|\le C_P\left\|\frac{e_u^{n+1}-e_u^n}{\Delta t}\right\|_0\left|\upsilon_h\right|_1+\upsilon\left\|\nabla e_u^{n+1/2}\right\|_0\left|\upsilon_h\right|_1.\ (20)$$

Using the definition of $b(\cdot;\cdot,\cdot)$ and Green's formula, $\nabla\cdot u_h^{n+1/2}=\nabla\cdot u=0$, we get

$$\left|R_4\right|\le\left|b\left(u_h^{n+1/2};e_u^{n+1/2},\upsilon_h\right)\right|+\left|b\left(e_u^{n+1/2};u^{n+1/2},\upsilon_h\right)\right|=$$
$$=\left|\left(u_h^{n+1/2}\cdot\nabla\upsilon_h,e_u^{n+1/2}\right)\right|+\left|\left(e_u^{n+1/2}\cdot\nabla\upsilon_h,u^{n+1/2}\right)\right|\le$$
$$\le C\left\|u_h^{n+1/2}\right\|_{0,\infty}\left\|e_u^{n+1/2}\right\|_0\left|\upsilon_h\right|_1+C\left\|u^{n+1/2}\right\|_{0,\infty}\left\|e_u^{n+1/2}\right\|_0\left|\upsilon_h\right|_1\le\qquad(21)$$
$$\le C\max\left(\left\|u^{n+1/2}\right\|_{0,\infty},\left\|u_h^{n+1/2}\right\|_{0,\infty}\right)\left\|e_u^{n+1/2}\right\|_0\left|\upsilon_h\right|_1;$$

$$\left|R_5+R_6\right|\le C\alpha\left\|\nabla_H\left(id-P_H\right)e_u^{n+1/2}\right\|_0\left|\upsilon_h\right|_1+$$
$$+C\alpha\left\|\nabla_H\left(id-P_H\right)u^{n+1/2}\right\|_0\left|\upsilon_h\right|_1;\qquad(22)$$

$$\left|R_7\right|\le C\Delta t^{3/2}\left|\upsilon_h\right|_1\times$$
$$\times\left(\left\|u_{ttt}\right\|_{L^2\left(t_n,t_{n+1};L^2(\Omega)\right)}+\left\|u_{tt}\right\|_{L^2\left(t_n,t_{n+1};H^2(\Omega)\right)}+\right.$$
$$\left.+\left\|p_{tt}\right\|_{L^2\left(t_n,t_{n+1};H^1(\Omega)\right)}\right).\qquad(23)$$

From (21–23) and Theorem 4.2 and the discrete LBB condition we get

$$\Delta t\sum_{i=0}^{N-1}\frac{\left|\left(\nabla\cdot\upsilon_h,\gamma^{i+1/2}\right)\right|^2}{\left|\upsilon_h\right|_1^2}\le\Delta t\sum_{i=0}^{N-1}\left(\left\|\frac{e_u^{i+1}-e_u^i}{\Delta t}\right\|_0^2+\upsilon\left\|\nabla e_u^{i+1/2}\right\|_0^2\right)+$$
$$+C\Delta t\sum_{i=0}^{N-1}\left(\left(\max\left(\left\|u^{i+1/2}\right\|_{0,\infty},\left\|u_h^{i+1/2}\right\|_{0,\infty}^2\right)\left\|e_u^{i+1/2}\right\|_0^2\right)+\right.$$
$$\left.+\alpha\left\|\nabla\left(id-P_H\right)e_u^{i+1/2}\right\|_0^2+\alpha\left\|\nabla\left(id-P_H\right)u^{i+1/2}\right\|_0^2\right).$$

We get our final conclusion by triangle inequality.

A direct result from Theorem 4.3 is.

By corollary 33 under the condition of Theorem 4.2, we have

$$\Delta t\sum_{i=0}^{N-1}\left\|p^{i+1/2}-p_h^{i+1/2}\right\|_0^2\le$$
$$\le C\left(1+\Delta t^{-2}+\max\left(1,h^{-d}E_p\right)\right)E_p+Ch^{2r},$$

where

$$E_p=\upsilon h^{2r}+\alpha h^{2r}+\alpha^{-1}h^{2r+2}+H^{-2}h^{2r+2}+\alpha H^{2l}+\Delta t^4.$$

Remark. The choice of $l=r$, $\alpha=Ch^{\frac{2(r+1)}{r+2}}$, $H=Ch^{\frac{r+1}{r+2}}$, $\Delta t=Ch^{\frac{(r+1)^2}{2(r+2)}}$ admits

$$\max_{1\le i\le N}\left\|u^i-u_h^i\right\|_0^2\le C\left(\upsilon h^{2r}+h^{\frac{2(r+1)^2}{(r+2)}}+\Delta t^4\right);\qquad(24)$$

$$\Delta t\sum_{i=0}^{N-1}\left\|p^{i+1/2}-p_h^{i+1/2}\right\|_0^2\le C\left(\upsilon h^{2r}+h^{\frac{2(r+1)^2}{(r+2)}}+\Delta t^4\right)\Delta t^{-2}+Ch^{2r}.$$

We notice that in semi-discrete SD [4] and semi-discrete CIP methods for the equal-order elements, we have

$$\max_{0\le t\le T}\left\|u-u_h\right\|_0^2\le C\left(\upsilon h^{2r}+h^{2r+1}\right).\qquad(25)$$

So (24) is almost comparable with (25). The choice of $l=r-1$, $\alpha=Ch^2$, $H=Ch$, $\Delta t=Ch$ admits

$$\max_{1\le i\le N}\left\|u^i-u_h^i\right\|_0^2\le C\left(h^{2r}+\Delta t^4\right);$$

$$\Delta t\sum_{i=0}^{N-1}\left\|p^{i+1/2}-p_h^{i+1/2}\right\|_0^2\le C\left(h^{2r}+\Delta t^4\right)\Delta t^{-2}+Ch^{2r}.$$

Numerical results. In all experiments, the algorithms are implemented using public domain finite element software Freeefem++cs.

Convergence study. Let Ω be the unit square in $\mathbb{R}^2$, $T = 1$, $\mathrm{Re} = 10^3$ and $\mathrm{Re} = 10^8$. We use the example for which the true solution is

$$\begin{cases} u_1 = -\cos(\pi x)\sin(\pi y)\exp(-2\pi^2 t/\mathrm{Re}) \\ u_2 = \sin(\pi x)\cos(\pi y)\exp(-2\pi^2 t/\mathrm{Re}) \\ p = -0.25(\cos(2\pi x) + \cos(2\pi y))\exp(-4\pi^2 t/\mathrm{Re}) \end{cases},$$

which is the Green-Taylor vortex. It was used as a numerical test.

In computations, the operator P_H is defined as: for any $u \in L^2(\Omega)$, we are seeking $P_H u \in L_H$ so that

$$(\nabla_H u, \nabla_H v) = (\nabla_H P_H u, \nabla_H v),\ \forall v \in L_H. \qquad (26)$$

Notice (26) is well-posed if $\|\nabla_H v\|_0$ is a norm on L_H, this can be satisfied by choosing $L_H = P_2(\mathcal{J}_H) \cap L_0^2(\Omega)$, $L_H = P_1(\mathcal{J}_h) \cap L_0^2(\Omega)$, $L_H = P_1^{nc}(\mathcal{J}_h) \cap L_0^2(\Omega)$. According to (26), the stabilization term in (5) can be reduced to

$$S_h(u_h^{n+1/2}, v_h) = \alpha(\nabla_H (id - P_H) u_h^{n+1/2}, \nabla_H v_h).$$

The type of computational meshes is presented in Fig. 1. We consider III cases:

SV-VMS I

$$V_h = P_2(\mathcal{J}_h^1) \cap V, Q_h = P_1^{dc}(\mathcal{J}_h^1) \cap Q, L_H = P_1(\mathcal{J}_h^2);$$

SV-VMS II

$$V_h = P_2(\mathcal{J}_h^1) \cap V, Q_h = P_1^{dc}(\mathcal{J}_h^1) \cap Q, L_H = P_1(\mathcal{J}_h^1);$$

SV-VMS III

$$V_h = P_2(\mathcal{J}_h^1), \cap V Q_h = P_1^{dc}(\mathcal{J}_h^1) \cap Q, L_H = P_1^{nc}(\mathcal{J}_h^1),$$

compared with the VMS method with Taylor-Hood elements,

$$\text{TH-VMS: } V_h = P_2(\mathcal{J}_h^2), \quad Q_h = P_1(\mathcal{J}_h^2),$$

with a stabilized term

$$S_h(u_h^{n+1/2}, v_h) = \alpha((id - \Pi_h)\nabla u_h^{n+1/2}, (id - \Pi_h)\nabla v_h),$$

where Π_h is a L^2-projection onto $P_1(\mathcal{J}_h^2)$. In this section we let

$$\Delta t = 0.01, \alpha = 0.1h^2.$$

The symbols $r_{L(u)}$, $r_{H(u)}$ and $r_{L(p)}$ stand for the convergence rate of $\|u - u_h\|_0$, $\|u - u_h\|_1$ and $\|p - p_h\|_0$ and $E_{L(u)}$, the errors $E_{H(u)}$, E_{div} and $E_{L(p)}$ are represented by

$$E_{L(u)} = \left(\Delta t \sum_{i=0}^{N-1} \left\|u^{i+1} - u_h^{i+1}\right\|_0^2\right)^{1/2};$$

$$E_{H(u)} = \left(\Delta t \sum_{i=0}^{N-1} \left|u^{i+1} - u_h^{i+1}\right|_1^2\right)^{1/2};$$

$$E_{\mathrm{div}} = \left(\Delta t \sum_{i=0}^{N-1} \left\|\nabla \cdot (u^{i+1} - u_h^{i+1})\right\|_0^2\right)^{1/2};$$

$$E_{L(p)} = \left(\Delta t \sum_{i=0}^{N-1} \left\|p^{i+1/2} - p_h^{i+1/2}\right\|_0^2\right)^{1/2}.$$

The results show that our methods have better numerical performances than the VMS method. At $\mathrm{Re} = 10^3$, the numerical solutions of our method preserve the incompressibility and is by 50–60 % more accurate than the TH-VMS method. At $\mathrm{Re} = 10^8$, the approximation solutions of our method is much more accurate than the solution by the TH-VMS method.

Confined cavity flow. We calculate the confined cavity flow for Re = 400. The computation domain is $\Omega = [0, 1] \times [0, 1]$. The boundary condition is: at the top boundary $(u_1, u_2) = (1, 0)$, at other three boundaries $(u_1, u_2) = (0, 0)$. The type of computational mesh is still as $\mathcal{J}_h^1$ in Fig. 2 with 6144 elements. We use the following time-step sizes: for $0 \le t \le 2$, $\Delta t = 0.1$; for $2 < t \le 4$, $\Delta t = 0.2$; for $4 < t \le 8$, $\Delta t = 0.4$; for $8 < t \le 20$, $\Delta = 0.8$; for $20 < t \le 36$, $\Delta t = 16$. We show the contour lines when Re = 400, $t = 2, 4, 8, 36$ in Fig. 2–4. Our results are in agreement with those in [4].

Flow around a cylinder.

We consider the benchmark problem ‘flow around a cylinder’ to test our numerical schemes. This is a well-known benchmark usage. The domain with meshes is presented in Fig. 5, where the origin of the cylinder is at $(x, y) = (0.2, 0.2)$ and the radius is equal to 0.15. The time-dependent inflow profile is

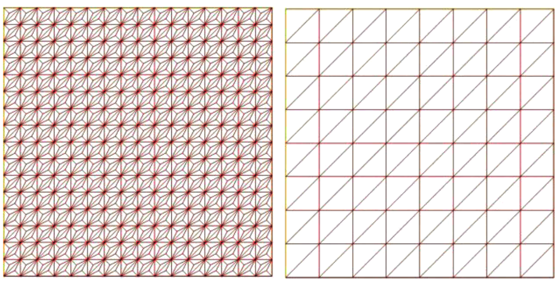

Fig. 1. The computational meshes

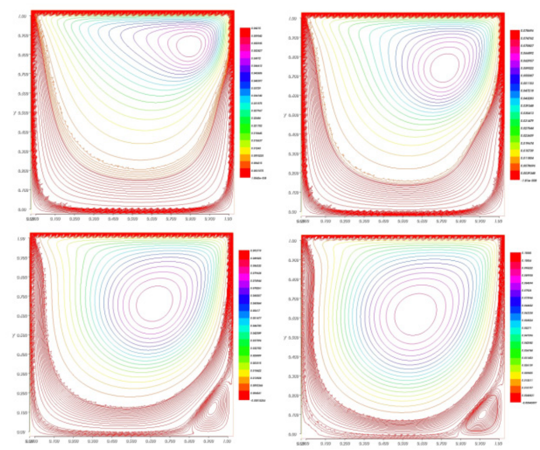

Fig. 2. SV-VMS I

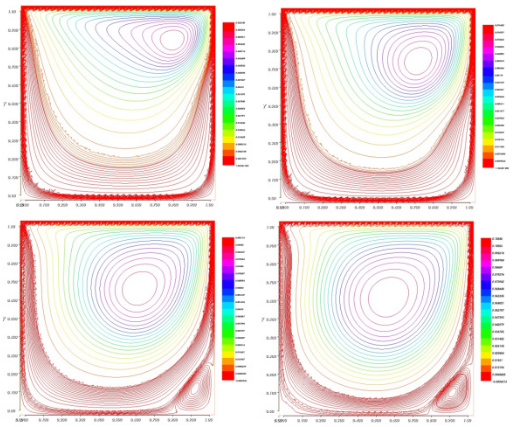

Fig. 3. SV-VMS II

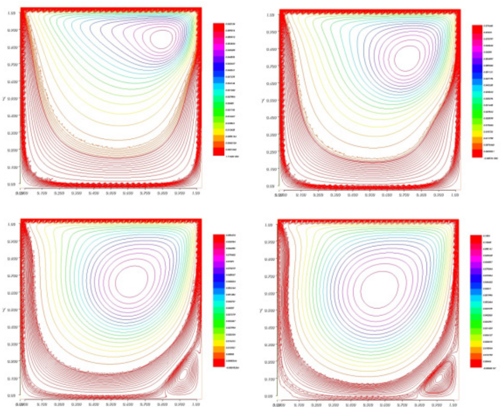

Fig. 4. SV-VMS III

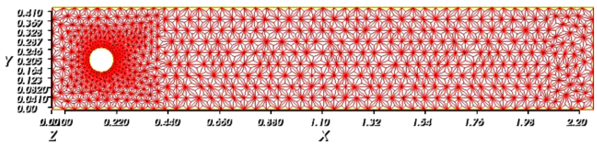

Fig. 5. The computational mesh

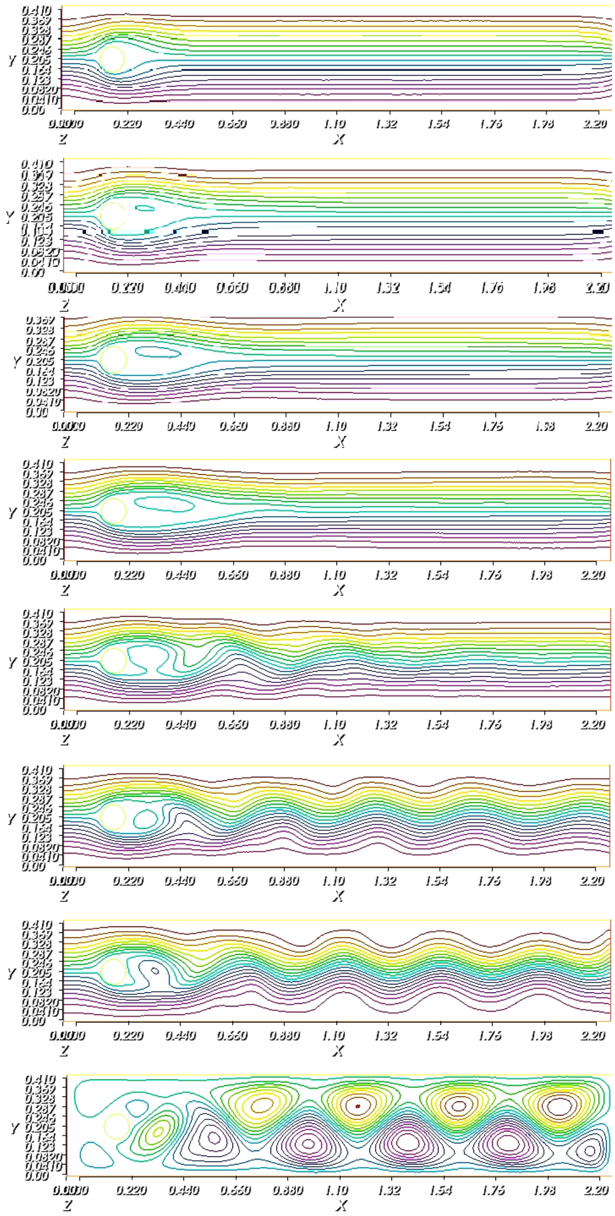

Fig. 6. The streamline

$$u_1(0,y,t)=u_1(2.2,y,t)=\frac{6}{0.41^2}\sin\frac{(\pi t)}{8}y(0.41-y);$$

$$u_2(0, y, t) = u_2(2.2, y, t) = 0.$$

Nonslip conditions are prescribed at the other boundaries. Computation is performed for Re = 1000 and the external force $f = 0$ using the scheme SV-VMS III. The contour lines of speed and streamline are plotted in Fig. 6 with a time step $\Delta t = 0.0025$. The results agree with those.

Conclusions. In this paper, we propose a new fully discrete VMS method using SV elements for the unsteady Navier-Stokes with high Reynolds number. This method features good stability. It preserves the incompressibility and it has error estimates which do not depend on the viscosity. We have noticed that the mapping P_H can be chosen other than (26), as long as it is well-defined which satisfies (4). The fact that L_H can be chosen as $P_1^{nc}(\mathcal{J}_h)\cap L_0^2(\Omega)$ is also very interesting. A linear treatment of our new method can be done by replacing $b\left(u_h^{n+1/2};u_h^{n+1/2},v_h\right)$ into $b\left(\frac{3}{2}u_h^n-\frac{1}{2}u_h^{n-1};u_h^{n+1/2},v_h\right)$ in (5), the analysis of this linear scheme is almost the same as ours in this paper.

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Мета. Велика кількість досліджень присвячена використанню варіаційних багатомасштабних (VMS) методів для рішення нестисливих течій. Цей аналіз відрізняється застосуванням так званого першого або другого флуктуаційного оператора. З іншої сторони, методи VMS використовуються для рішення нестаціонарних нестисливих течій. Отримані оцінки похибки, що залежать від наведеного числа Рейнольдса. З іншого боку, за допомогою методів дифузійної стабілізації (SD) та безперервних штрафних функцій (CIP) раніше отримані оцінки, що не залежать від числа Рейнольдса. Таким чином, ми хочемо отримати ті самі або схожі результати з використанням VMS методів.

Методика. Нами запропоновано повністю дискретний стабілізований метод для нестаціонарних рівнянь Нав›е-Стокса (NSE) з великим числом Рейнольдса. Ми використовуємо різницю Кранка-Ніколсона у часі та елементи Скотта-Вогеліуса (SV) у просторі для збереження нестисливості. Конвективні ефекти стабілізуються шляхом додавання нової проекції на основі VMS-елемента. Доведені стійкість і збіжність наближеного рішення. Оцінки похибки не залежать від числа Рейнольдса, а, відповідно, й для нестисливих рівнянь Ейлера, за умови, що точне рішення є гладким.

Результати. Доведена стійкість і збіжність рішення наближення. Оцінки похибок не залежать від числа Рейнольдса, а, відповідно, й для нестисливих рівнянь Ейлера, за умови, що точне рішення є гладким. Цей метод має гарну стабільність. Він зберігає нестисливість і має оцінки похибки, що не залежать від в'язкості.

Наукова новизна. У цій статті ми пропонуємо новий повністю дискретний VMS-метод з використанням SV-елементів для нестаціонарних рівнянь Нав›е-Стокса з високим числом Рейнольдса. Нестискуваність зберігається за допомогою елементів Скотта-Вогеліуса, конвективні ефекти стабілізуються шляхом додавання нової проекції на основі варіаційного багатомасштабного елемента.

Практична значимість. Чисельні експерименти показують, що запропонований метод є дуже ефективним для нестисливих течій з високим числом Рейнольдса. Вони також підтверджують, що даний метод зберігає нестисливість.

Ключові слова: *нестаціонарні рівняння Нав'е-Стокса з високим числом Рейнольдса, елементи Скотта-Вогеліуса, нестисливість, конвективні ефекти, різниця Кранка-Ніколсона, варіаційний багатомасштабний метод*

Цель. Многие исследования посвящены использованию вариационных многомасштабных (VMS) методов для решения несжимаемых потоков. Этот анализ отличается применением так называемого первого или второго флуктуационного оператора. С другой стороны, методы VMS используются для решения нестационарных несжимаемых течений. Получены оценки погрешности, зависящие от приведенного числа Рейнольдса. С другой стороны, с помощью методов диффузионной стабилизации (SD) и непрерывных штрафных функций (CIP) ранее получены оценки, которые не зависят от числа Рейнольдса. Таким образом, мы хотим получить те же или сходные результаты с использованием VMS-методов.

Методика. Нами предложен полностью дискретный стабилизированный метод для нестационарных уравнений Навье-Стокса (NSE) с большим числом Рейнольдса. Мы используем разность Кранка-Николсона во времени и элементы

Скотта-Вогелиуса (SV) в пространстве для сохранения несжимаемости. Конвективные эффекты стабилизируются путем добавления новой проекции на основе VMS-элемента. Доказаны устойчивость и сходимость приближенного решения. Оценки погрешности не зависят от числа Рейнольдса, а, следовательно, и для несжимаемых уравнений Эйлера, при условии, что точное решение является гладким

Результаты. Доказана устойчивость и сходимость решения приближения. Оценки погрешностей не зависят от числа Рейнольдса, а, следовательно, и для несжимаемых уравнений Эйлера, при условии, что точное решение является гладким. Этот метод имеет хорошую стабильность. Он сохраняет несжимаемость и имеет оценки погрешности, которые не зависят от вязкости.

Научная новизна. В этой статье мы предлагаем новый полностью дискретный VMS-метод с использованием SV-элементов для нестационарных уравнений Навье-Стокса с высоким числом Рейнольдса. Несжимаемость сохраняется с помощью элементов Скотта-Вогелиуса, конвективные эффекты стабилизируются путем добавления новой проекции на основе вариационного многомасштабного элемента.

Практическая значимость. Численные эксперименты показывают, что предложенный метод очень эффективен для несжимаемых течений с высоким числом Рейнольдса. Они также подтверждают, что данный метод сохраняет несжимаемость

Ключевые слова: *нестационарные уравнения Навье-Стокса с высоким числом Рейнольдса, элементы Скотта-Вогелиуса, несжимаемость, конвективные эффекты, разность Кранка-Николсона, вариационный многомасштабный метод*

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